

Bose gas to Bose–Einstein Condensate by the Phase Transition of the Klein–Gordon equation

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We rewrite the complex Klein-Gordon (KG) equation with a mexican-hat scalar field potential in a thermal bath with one loop contribution as a new Gross-Pitaevskii-like equation. We interpret it as a charged and finite temperature generalization of the Gross-Pitaevskii equation. We find its hydrodynamic version as well and using it, we derive the corresponding thermodynamics. We obtain a generalized first law for a charged Bose-Einstein Condensate (BEC). We translate the breaking of the $U(1)$ local symmetry of the KG field into the new version of the Gross-Pitaevskii equation and demonstrate that this symmetry breaking corresponds to a phase transition of the gas into a BEC and show the conditions for condensation and/or phase transition for which this system naturally becomes superfluid and/or superconductor.

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INTRODUCTION

Since its observation with the help of magnetic traps [1], the phenomenon of Bose–Einstein condensation has spurred an enormous amount of works on the theoretical and experimental realms associated to this topic. The principal interest in the study on Bose–Einstein condensation is its interdisciplinary nature. From the thermodynamic point of view, this phenomenon can be interpreted as a phase transition, and from the quantum mechanical point of view as a matter wave coherence arising from overlapping de Broglie waves of the atoms, in which many of them condense to the ground state of the system. In quantum field theory, this phenomenon is related to the spontaneous breaking of a gauge symmetry [2]. Symmetry breaking is one of the most essential concepts in particle theory and has been extensively used in the study of the behavior of particle interactions in many theories [3]. Phase transitions are changes of state, related with changes of symmetries of the system. The analysis of Symmetry breaking mechanisms have turn out to be very helpful in the study of phenomena associated to phase transitions in almost all areas of physics. Bose-Einstein Condensation is one topic of interest that uses in an extensive way the Symmetry breaking mechanisms [2], its phase transition associated with the condensation of atoms in the state of lowest energy and is the consequence of quantum, statistical and thermodynamical effects.

On the other hand, the results from finite temperature quantum field theory [4, 5] raise important challenges about their possible manifestation in condensed matter systems. By investigating the massive Klein–

Gordon equation (KG), in [6] we were able to show, that the KG equation can simulate a condensed matter system. In [6] it was shown that the KG equation with a self interacting scalar field (SF) in a thermal bath reduces to the Gross–Pitaevskii equation in the no-relativistic limit, provided that the temperature of the thermal bath is zero. Thus, the KG equation reduces to a generalized relativistic, Gross–Pitaevskii (GP) equation at finite temperature. But a question remains open. The KG equation with a self interacting SF potential defines a symmetry breaking temperature, at which the system experiments a phase transition. However, this phase transition does not necessarily means a condensation of the particles of the system.

In the present work we study the complex KG equation with a Mexican hat SF potential in a thermal bath. The idea is very simple, the KG equation up to one loop in perturbations is able to explain the phase transition of a SF, like the Higgs field, close to the moment of the phase transition, when the KG equation breaks the $U(1)$ symmetry of the corresponding Lagrangian. On the other hand, the Gross-Pitaevskii equation is able to explain the behavior of a BEC at zero temperature. In this work we wonder if the KG equation in a thermal bath is able to generalize the Gross-Pitaevskii equation in a region close to the phase transition of the Bose gas into a BEC, because the KG equation contains the information of the temperature of the bath in the SF potential. In base of this idea, we rewrite the complex KG equation into a Gross-Pitaevskii-like equation and interpret it as a Gross-Pitaevskii one at finite temperature. In order to see if this equation can explain the phase transition of a Bose gas into a BEC we write its corresponding thermodynamics, deriving a corresponding first law for the Bose gases. The only difference we find here with respect to the traditional first law of the thermodynamics is a term where the quantum mechanical character of the KG equation is present. On the other hand, a phase transition does

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no necessarily means gas condensation. Therefore we obtain the temperature and the conditions of condensation of this Bose gas. At the end of the work we qualitatively see under which conditions the Bose gas becomes superfluid or/and superconductor.

GAUGE SYMMETRY BREAKING

We start with a model having a local U(1) symmetry given by the lagrangian

$$\mathcal{L} = -(\nabla_\mu \Phi^* + ieA_\mu \Phi^*)(\nabla^\mu \Phi - ieA^\mu \Phi) - V(|\Phi|) - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} \quad (1)$$

where $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$ is the Maxwell tensor and the SF potential V is the easiest case of a double-well interacting Mexican-hat potential for a complex SF $\Phi(\vec{x}, t)$, interacting inside a thermal bath in a reservoir that can have interaction with its surroundings up to one loop of correction, that goes as

$$V(\Phi) = -\hat{m}^2 \Phi \Phi^* + \frac{\hat{\lambda}}{2} (\Phi \Phi^*)^2 + \frac{\hat{\lambda}}{4} k_B^2 T^2 \Phi \Phi^* - \frac{\pi^2 k_B^4}{90 \hbar^2 c^2} T^4. \quad (2)$$

where $\hat{m}^2 = m^2 c^2 / \hbar^2$ is the scalar particle mass, $\hat{\lambda} = \lambda / (\hbar^2 c^2)$ is the parameter describing the interaction, k_B is the Boltzmann's constant, \hbar is the Planck's constant, c is the speed of light and T is the temperature of the thermal bath, this result includes both quantum and thermal contributions.

From here the interpretation of the different quantities of the system is clear, Φ is the KG version of the Gross-Pitaevskii function Ψ (see below), the Maxwell field is an electromagnetic field in the system, λ is the self-interaction parameter related with the scattering length of the system and T is the temperature of the thermal bath where the system lie.

The dynamics of a SF is governed by the KG equation, it is the equation of motion of a field composed of spinless particles. In order to confine the BEC experimentally one needs to add an external field controlled by hand in order to cause the condensation, like a laser for example. In order to do this, we will add an external field ϕ that will interact with the SF to first order, such that the KG equation will be given by

$$\square_E^2 \Phi - \frac{dV}{d\Phi^*} - 2\hat{m}^2 \phi \Phi = 0, \quad (3)$$

where for a charged field the D'Ambertian operator is given by,

$$\square_E^2 \equiv (\nabla_\mu - ieA_\mu)(\nabla^\mu - ieA^\mu) \quad (4)$$

where A_μ is the electromagnetic four potential. Observe that we can rewrite the D'Ambertian as

$$\square_E^2 = (\nabla - 2ie\mathbf{A}) \cdot \nabla - \left(\frac{1}{c} \frac{\partial}{\partial t} - 2ie\varphi\right) \frac{1}{c} \frac{\partial}{\partial t} - ie\nabla_\mu A^\mu - e^2 A_\mu A^\mu \quad (5)$$

where we have written the electromagnetic four potential as $A_\mu = (\mathbf{A}, \varphi)$. In what follows we will use the Lorentz gauge $\nabla_\mu A^\mu = 0$. It is convenient to consider the total (effective) potential V adding the external one and the term $e^2 A_\mu A^\mu = e^2 A^2$ to the potential, such that

$$V_T(\Phi) = -\frac{m^2 c^2}{\hbar^2} \Phi \Phi^* + \frac{\lambda}{4 \hbar^2 c^2} k_B^2 T^2 \Phi \Phi^* - e^2 A^2 \Phi \Phi^* + \frac{\lambda}{2 \hbar^2 c^2} (\Phi \Phi^*)^2 - \frac{\pi^2 k_B^4}{90 \hbar^2 c^2} T^4 + \frac{2m^2 c^2}{\hbar^2} \phi \Phi \Phi^*. \quad (6)$$

where we see that the parameter e is the coupling constant between the electromagnetic and the SFs. In terms of the effective potential V_T the KG equation now can be written as

$$\square^2 \Phi - ieA^\mu \nabla_\mu \Phi - \frac{dV_T}{d\Phi^*} = 0, \quad (7)$$

where now $\square^2 = \nabla^\mu \nabla_\mu$. The Maxwell equations also read

$$\begin{aligned} \nabla^\mu F_{\mu\nu} &= -j_\nu \\ &= ie(\Phi^* \nabla_\nu \Phi - \Phi \nabla_\nu \Phi^*) + 2e^2 \Phi \Phi^* A_\nu. \end{aligned} \quad (8)$$

We can define an effective mass by

$$m_{eff} c^2 = \sqrt{m^2 c^4 + \hat{e}^2 A^2}. \quad (9)$$

where $\hat{e} = e\hbar c$. For the V_T potential (6), the critical temperature T_c^{SB} where the minimum of the potential $\Phi = 0$ becomes a maximum and at which the symmetry is broken is,

$$k_B T_c^{SB} = \frac{2c^2}{\sqrt{\lambda}} \sqrt{m_{eff}^2 - 2m^2 \phi}. \quad (10)$$

Potential (6) has a minimum in $\Phi = 0$ when the temperature $T > T_c^{SB}$. If $T < T_c^{SB}$, the point $\Phi = 0$ becomes a maximum and potential (6) has two minima in

$$\begin{aligned} R_{min} &= \pm \sqrt{\frac{1}{\lambda} (m^2 c^4 + \hat{e}^2 A^2 - \frac{\lambda}{4} k_B^2 T^2 - 2m^2 c^4 \phi)} \\ &= \pm \frac{k_B}{2} \sqrt{(T_c^{SB})^2 - T^2} \end{aligned} \quad (11)$$

being $\Phi = R e^{i\Theta}$. In the maximum $\Phi = 0$ the second derivative of the potential V_T with respect to the SF reads

$$\begin{aligned} V_{T,\Phi\Phi} &= - \left(\hat{m}^2 + e^2 A^2 - \frac{\hat{\lambda}}{4} k_B^2 T^2 - 2\hat{m}^2 \phi \right) \\ &= - \frac{\hat{\lambda}}{4} k_B^2 ((T_c^{SB})^2 - T^2) \\ &= - \frac{(T_c^{SB})^2 - T^2}{(T_c^{SB})^2} (\hat{m}_{eff}^2 - 2\hat{m}^2 \phi) c^4. \end{aligned} \quad (12)$$

In what follows we will re-write the KG equations in order to interpret the KG equation as a Gross-Pitaevskii one.

THE GENERALIZED GROSS-PITAEVSKII EQUATION

Now for the SF we perform the transformation

$$\Phi = \Psi e^{-i\hat{m}ct},$$

In terms of the complex function Ψ , the KG equation (3) now reads,

$$\begin{aligned} i\hbar c \dot{\Psi} + \frac{\hbar^2}{2m} \square_E^2 \Psi - \frac{\lambda}{2mc^2} |\Psi|^2 \Psi - [mc^2(\phi - 1) + ec\phi] \Psi \\ - \frac{\lambda k_B^2 T^2}{8mc^2} \Psi = 0, \end{aligned} \quad (13)$$

where we have kept just the equation for the Ψ part, the complex conjugate can be described in the same way. The notation used is $\dot{} = 1/c \partial/\partial t$ (13) is the KG equation (3) or (7) rewritten in terms of the function Ψ and temperature T . This equation is an exact equation defining the field $\Psi(\mathbf{x}, t)$, where ϕ defines the external potential acting on the system and the term with λ represents the self-interaction of the SF within the system. We will consider equation (13) as a generalization of the Gross-Pitaevskii equation for finite temperatures and relativistic particles. This is because when $T = 0$ and in the non-relativistic limit, $\square^2 \rightarrow \nabla^2$, eq. (13) becomes the Gross-Pitaevskii equation for Bose-Einstein Condensates (BEC), provided that $\lambda = 8\pi\hbar^2 c^2 a \kappa^2$, being a the s-wave scattering length [8]. The static limit of equation (13) is known as the Ginzburg-Landau equation. Observe that the temperature T and the external field ϕ must be manipulated from outside.

At this point it is noteworthy to mention the order of magnitude of the quantity (10), the associated critical temperature at which the symmetry of the system is broken. Assuming that the external potential ϕ and the additional field A^2 are time independent, i.e., that the system is in static thermal equilibrium, we can write $k_B T_c^{SB} \sim \frac{2mc^2}{\sqrt{\lambda}}$ in the center of the system. If we set $\lambda = 8\pi\hbar^2 c^2 a \kappa^2$ into (10), thus $T_c^{SB} \sim 8.21 \frac{\text{Joules}^{-1} \text{m}^{-3/2}}{\kappa} \left(\frac{m}{\text{gr}} \right) \sqrt{\frac{\text{cm}}{a}} \times 10^{62} \text{ K}$. For instance, in the case of ^{87}Rb , with a mass $m \sim 1.452 \times 10^{-22} \text{ gr}$ and a scattering length $a \sim 5.2 \times 10^{-7} \text{ m}$, the critical temperature of symmetry breakdown of this system is $T_c^{SB} \sim \frac{1.65}{\kappa} \times 10^{44} \text{ K}$, which depending on the value of κ , it could be very big. However, for example, if we set $\kappa \sim 5 \times 10^{50} \text{ Joules}^{-1} \text{ m}^{-3/2}$ (see also [9]), we obtain that $T_c^{SB} \sim 3 \times 10^{-7} \text{ K}$. Nevertheless, the density n does not depend on the value of κ , because $n = \kappa^2 k_B (T_c^{SB})^2 / 4(1 - (T/T_c^{SB})^2) \sim$

$\kappa^2 k_B (T_c^{SB})^2 / 4 = m^2 c^2 / (8\pi\hbar^2 a)$, and this quantity for the ^{87}Rb is $\sim 10^{36} / \text{cm}^3$. Multiplying this density times the mass, we obtain the corresponding mass density, which is $\sim 1.9 \times 10^{14} \text{ gr/cm}^3$. This mass density corresponds to the one of a nuclear atom or a neutron star. Furthermore, for this values the self-interaction parameter λ is very big $\lambda \sim 3.2 \times 10^{43}$. Let us suppose for a moment that neutrons can build cooper pairs in a neutron star. In that case, the mass of two neutrons is $m = 2 \times 1.6 \times 10^{-24} \text{ gr}$, the scattering length of the neutrons is $a \sim 10^{-11} \text{ cm}$. Thus, the critical temperature of symmetry breakdown of this system is $T_c^{SB} \sim \frac{8.31}{\kappa} 10^{44} \text{ K}$. If we set the typical temperature of collapse for a neutron star $T \sim 10^{12} \text{ K}$ as the temperature of the symmetry breakdown, we obtain that $\kappa \sim 10^{32} \text{ Joules}^{-1} \text{ m}^{-3/2}$. In this case $\lambda \sim 25$, the particle density $n \sim 3.2 \times 10^{37} / \text{cm}^3$ and the mass density $\sim 10^{14} \text{ gr/cm}^3$, which is again of the order of magnitude of neutron stars (see [10]). Other interesting example is BECs as Dark Matter [11]. In this case the mass of the SF could be of the order of the axion mass $m \sim 0.1 \text{ eV} = 1.78 \times 10^{-34} \text{ gr}$. Here there are two cases, if the SF is an axion [12], the self-interaction parameter is $\lambda \sim 10^{15}$ [13], for this value of the mass and the self-interacting parameter, the critical temperature is $T_c^{SB} \sim 7 \times 10^{-5} \text{ K} = 6 \times 10^{-9} \text{ eV}$. The mass density is as expected $\rho \sim 10^{-45} \text{ gr/cm}^3$. On the other hand, if the SF are Scalar Field Dark Matter particles [14], the mass is again $m \sim 0.1 \text{ eV} = 1.78 \times 10^{-34} \text{ gr}$, but the self-interaction parameter is $\lambda \sim 10^{-6}$. For that the critical temperature is $T_c^{SB} \sim 2.3 \times 10^6 \text{ K} = 200 \text{ eV}$ for a mass density of the order of the critical density of the universe $\rho \sim 10^{-29} \text{ gr/cm}^3$.

THE HYDRODYNAMICAL VERSION

In what follows we transform the generalized Gross-Pitaevskii equation (13) into its analogous hydrodynamical version, [15, 16], for this purpose the ensemble wave function Ψ will be represented in terms of a modulus n and a phase S as,

$$\kappa \Psi = \sqrt{n} e^{iS}. \quad (14)$$

where the phase $S(\mathbf{x}, t)$ is taken as a real function. As usual this phase will define the velocity. Here we will interpret $n(\mathbf{x}, t)$ as the number density of particles in the symmetry broken state, such that $\kappa^2 |\Psi|^2 = \kappa^2 \Psi \Psi^* = n$, where κ is the scale of the system, which is to be determined by an experiment, being both, S and n , functions of time and position. So, from this interpretation we have that when the KG equation oscillates around the $\Phi = 0$ minimum, the number of particles in the ground state is zero, $n = 0 = \rho$. Below the critical temperature T_c^{SB} , close to the second minimum, $R_{min}^2 = k_B^2 ((T_c^{SB})^2 - T^2) / 4$, the density will oscillate

around $n = \kappa^2 k_B^2 ((T_c^{SB})^2 - T^2)/4$ as can be seen by equation (14). In order to see this more clear, we perform the Madelung transformation (14) in the generalized Gross-Pitaevskii equation (13).

From (13) and (14), separating real and imaginary parts we obtain

$$\begin{aligned} c\dot{n} + \frac{\hbar}{m}n [\Box^2 S - e(\nabla \cdot \mathbf{A} - \dot{\varphi})] + \\ \frac{\hbar}{m} \left((\nabla S - e\mathbf{A}) \cdot \nabla n - (\dot{S} - e\varphi)\dot{n} \right) = 0, \quad (15a) \\ \frac{\hbar c}{m} (\dot{S} - e\varphi) + \frac{\lambda}{2m^2 c^2 \kappa^2} n + c^2(\phi - 1) + \frac{\lambda}{8m^2 c^2} k_B^2 T^2 + \\ \frac{\hbar^2}{m^2} \left(\frac{\Box^2 \sqrt{n}}{\sqrt{n}} \right) + \frac{\hbar^2}{2m^2} \left((\nabla S - e\mathbf{A})^2 - (\dot{S} - e\varphi)^2 \right) = 0. \quad (15b) \end{aligned}$$

Taking the gradient of (15b) and using the definitions of the fluxes

$$\mathbf{j} = \frac{2en}{\kappa^2} (\nabla S - e\mathbf{A}) \quad (16a)$$

$$j = \frac{2en}{\kappa^2} (\dot{S} - e\varphi) \quad (16b)$$

$$j_\mu = (\mathbf{j}, j - \frac{2en}{\kappa^2} \dot{n}) \quad (16c)$$

and the velocity

$$\mathbf{v} \equiv \frac{\hbar}{m} (\nabla S - e\mathbf{A}) \quad (17)$$

equations (15) can be rewritten as,

$$\begin{aligned} \dot{n} + \nabla \cdot (n\mathbf{v}) - \frac{\hbar \kappa^2}{2mec} \dot{j} = 0, \quad (18a) \\ \dot{\mathbf{v}} + (\mathbf{v} \cdot \nabla) \mathbf{v} - \frac{\hbar}{m} e (c\mathbf{E} + \mathbf{v} \times \mathbf{B}) = \\ -c^2 \nabla \phi - \frac{\lambda}{m^2 c^2 \kappa^2} \nabla n - \frac{\hbar^2}{m^2} \nabla \left(\frac{\nabla^2 \sqrt{n}}{\sqrt{n}} \right) + \\ \frac{\hbar^2}{2m^2} \nabla (\dot{S} - e\varphi)^2 + \frac{\hbar^2}{m^2} \nabla \left(\frac{\partial_t^2 \sqrt{n}}{\sqrt{n}} \right) - \frac{\lambda k_B^2}{4m^2} T \nabla T \quad (18b) \end{aligned}$$

where $\mathbf{E} = -1/c \partial \mathbf{A} / \partial t + \nabla \cdot \varphi$ and $\mathbf{B} = \nabla \times \mathbf{A}$ respectively are the electric and the magnetic field vectors. Notice that in (18b) \hbar enters on the right-hand side through the term containing the gradient of n . This term is usually called the 'quantum pressure' and is a direct consequence of the Heisenberg uncertainty principle, it reveals the importance of quantum effects in interacting gases. Multiplying by n , (18b) can be written as:

$$n\dot{\mathbf{v}} + n(\mathbf{v} \cdot \nabla) \mathbf{v} = n\mathbf{F}_E + n\mathbf{F}_\phi - \nabla p + n\mathbf{F}_Q + \nabla \sigma, \quad (19)$$

where $\mathbf{F}_E = \frac{e}{m} (c\mathbf{E} + \mathbf{v} \times \mathbf{B})$ is the electromagnetic force, $\mathbf{F}_\phi = -\nabla \phi$ is the force associated to the external potential ϕ , p can be seen as the pressure of the

SF gas that satisfies the equation of state $p = wn^2$, being $\omega = \lambda/(2m^2 c^2 \kappa^2)$ an interaction parameter. ∇p are forces due to the gradients of pressure, $\mathbf{F}_Q = -\nabla U_Q$ is the quantum force associated to the quantum potential, [8, 17],

$$U_Q = \frac{\hbar^2}{m^2} \left(\frac{\nabla^2 \sqrt{n}}{\sqrt{n}} \right), \quad (20)$$

and $\nabla \sigma$ is defined as

$$\begin{aligned} \nabla \sigma = \frac{\hbar^2}{2m^2} n \nabla (\dot{S} - e\varphi)^2 - \frac{1}{4} \frac{\lambda}{m^2} k_B^2 n T \nabla T \\ - \zeta \nabla (\ln n) + \frac{\hbar^2 n}{2m^2} \nabla \left(\frac{\ddot{n}}{n} \right), \quad (21) \end{aligned}$$

where the coefficient ζ is given by

$$\zeta = \frac{\hbar^2}{2m^2} \left[-\nabla \cdot (n\mathbf{v}) + \frac{\hbar \kappa^2}{2mec} \dot{j} \right].$$

Observe that using (18a) the term $\nabla (\ln n)$ can be written as

$$\nabla (\ln n) = -\nabla (\nabla \cdot \mathbf{v}) - \nabla [\nabla (\ln n) \cdot \mathbf{v}] + \frac{\hbar \kappa^2}{2mec} \nabla \left(\frac{1}{n} \dot{j} \right).$$

System (18) is the hydrodynamical representation to equation (13) and up to one constant, it is completely equivalent to (13).

THE NEWTONIAN (NON-RELATIVISTIC) LIMIT

Neglecting second order time derivatives and products of time derivatives we can simplify system (18). In this limit we arrive to the non-relativistic system of equations (18),

$$\dot{n} + \nabla \cdot (n\mathbf{v}) = 0, \quad (22a)$$

$$n\dot{\mathbf{v}} + n(\mathbf{v} \cdot \nabla) \mathbf{v} = n\mathbf{F}_E + n\mathbf{F}_\phi - \nabla p + n\mathbf{F}_Q + \nabla \sigma. \quad (22b)$$

Equation (22a) is the continuity equation, and (22b) is the equation for the momentum. Observe that this last one contains forces due to the external potential, to the gradient of the pressure, viscous forces due to the interactions of the condensate and forces due to the quantum nature of the equations. Quantity $\nabla (\ln n)$ plays a very important roll, in this limit it reads

$$\nabla (\ln n) = -\nabla (\nabla \cdot \mathbf{v}) - \nabla [\nabla (\ln n) \cdot \mathbf{v}].$$

Thus

$$\nabla \sigma = -\frac{1}{4} \frac{\lambda}{m} k_B^2 n T \nabla T - \zeta [\nabla (\nabla \cdot \mathbf{v}) + \nabla [\nabla (\ln n) \cdot \mathbf{v}]] \quad (23)$$

where now we have

$$\zeta = -\frac{\hbar^2}{2m^2} \nabla \cdot (n\mathbf{v}),$$

We interpret the function $\nabla\sigma$ as the viscosity of the system, it contains terms which are gradients of the temperature and of the divergence of the velocity and density (dissipative contributions). The measurement of the temperature dependence in this thermodynamical quantity at the phase transitions might reveal important information about the behavior of the gas due to particle interaction.

THE THERMODYNAMICS

In what follows we will derive the thermodynamical equations for the non-relativistic limit from the hydrodynamical representation. We can derive a conservation equation for a function α , starting with the relationship

$$(n\alpha)' = n\dot{\alpha} + \alpha\dot{n} \quad (24)$$

where α can take the values of ϕ , U_Q , etc., all of them fulfil equation (24). Using the continuity equation (22a) in (24) we obtain,

$$(n\alpha)' + \nabla \cdot (n\mathbf{v}\alpha) = -n\mathbf{v} \cdot \mathbf{F}_\alpha + n\dot{\alpha}.$$

Nevertheless, this procedure is not possible for σ because in general we do not know it explicitly, only in some cases it might be possible to integrate it.

Observe how the quantum potential U_Q also fulfills the following relationship

$$n\dot{U}_Q + \nabla \cdot (n\mathbf{v}_\rho) = 0, \quad (25)$$

which follows by direct calculation, and where we have defined the velocity density \mathbf{v}_ρ by

$$\mathbf{v}_\rho = \frac{\hbar^2}{4m^2} (\nabla \ln n),$$

which can be interpreted as a velocity flux due to the potential U_Q . Using the continuity equation (22a), equation (25) can be rewritten as

$$(nU_Q)' + \nabla \cdot (nU_Q\mathbf{v} + \mathbf{J}_\rho) + n\mathbf{v} \cdot \mathbf{F}_Q = 0 \quad (26)$$

where we have defined the quantum density flux

$$\mathbf{J}_\rho = n\mathbf{v}_\rho.$$

Equation (26) is another expression for the continuity equation of the quantum potential U_Q .

As we know, for the non-relativistic limit the total energy density of the system ϵ is the sum of the kinetic,

potential and internal energies [18], in this case we have an extra term U_Q due to the quantum potential,

$$\epsilon = \frac{1}{2}nv^2 + n\phi + nu + nU_Q + \psi_E \quad (27)$$

being u the inner energy of the system and

$$\psi_E = \frac{e}{m}(\varphi - \mathbf{v} \cdot \mathbf{A}) \quad (28)$$

the electromagnetic energy potential, defined in terms of the vector potential \mathbf{A} and the electric potential φ . Observe that ψ_E fulfills the continuity equation

$$(n\psi_E)' + \nabla \cdot (n\mathbf{v}\psi_E + \mathbf{j}_B) = n\dot{\psi}_E - n\mathbf{v} \cdot \mathbf{F}_E \quad (29)$$

being \mathbf{j}_B given by the continuity equation of the vector potential \mathbf{A}

$$\frac{\partial \mathbf{A}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{A} = -(\mathbf{A} \cdot \nabla)\mathbf{v} + \frac{m}{e}\mathbf{j}_B, \quad (30)$$

Then from (27) we have that u will satisfy the equation

$$(nu)' + \nabla \cdot \mathbf{J}_u - \nabla \cdot \mathbf{J}_\rho + n\dot{\phi} = -p\nabla \cdot \mathbf{v}, \quad (31)$$

being \mathbf{J}_u the energy current, given by a energy flux and a heat flux, \mathbf{J}_q ,

$$\mathbf{J}_u = nu\mathbf{v} + \mathbf{J}_q + \mathbf{J}_B - p\mathbf{v},$$

where $\nabla \cdot \mathbf{J}_q = \mathbf{v} \cdot (\nabla\sigma)$, and $\nabla \cdot \mathbf{J}_B = \mathbf{v} \cdot (n\mathbf{j}_B)$, expressions that as we can see are related in a direct way to the velocity and gradients of temperature in the condensate, and are the ones that show in an explicit way the temperature dependence of the thermodynamical equations. With these definitions at hand we have,

$$(nu)' + \nabla \cdot (n\mathbf{v}u + \mathbf{J}_q + \mathbf{J}_B - p\mathbf{v} - \mathbf{J}_\rho) + n\dot{\phi} = -p\nabla \cdot \mathbf{v}. \quad (32)$$

In order to find the thermodynamical quantities of the system in equilibrium (taking p as constant on a given volume), we restrict the system to the regime where the auto-interacting potential is constant in time, with this conditions at hand for (32) we have:

$$(nu)' + \nabla \cdot (n\mathbf{v}u + \mathbf{J}_q + \mathbf{J}_B - p\mathbf{v} - \mathbf{J}_\rho) = -p\nabla \cdot \mathbf{v} \quad (33)$$

From (33) we can have a straightforward interpretation of the terms involved in the phase transition. As always the first term will represent the change in the internal energy of the system, $-p\nabla \cdot \mathbf{v}$ is the work done by the pressure and $\nabla \cdot \mathbf{v}$ is related to the change in the volume, \mathbf{J}_q contains terms related to the heat generated by gradients of the temperature ∇T and dissipative forces due to viscous forces $\sim \nabla(\nabla \cdot \mathbf{v})$ and finally but most important we have an extra term, $\nabla \cdot \mathbf{J}_\rho$, due to gradients of the quantum potential (20).

Integrating this resulting expression on a close region, we obtain

$$\begin{aligned} \frac{d}{dt} \int nu \, dV + \oint (\mathbf{J}_q + \mathbf{J}_B + p\mathbf{v}) \cdot \mathbf{n} \, dS - \oint \mathbf{J}_\rho \cdot \mathbf{n} \, dS \\ = -p \frac{d}{dt} \int dV. \end{aligned}$$

Equation (33) is the continuity equation for the internal energy of the system and as usual, from here we have an expression that would describe the thermodynamics of the system in an analogous way as does the first law of thermodynamics, in this case for the KG equation or a BEC. This reads

$$dU = \hat{d}Q + \hat{d}A_Q + \hat{d}Q_B - p dV \quad (34)$$

where $U = \int nu \, dV$ is the internal energy of the system, [42], and as we can see, its change is due to a combination of heat Q added to the system and work done on the system. Furthermore, we have that

$$\frac{\hat{d}A_Q}{dt} = \frac{\hbar^2}{4m^2} \oint n(\nabla \ln n) \cdot \mathbf{n} \, dS = \oint n \mathbf{v}_\rho \cdot \mathbf{n} \, dS,$$

is the corresponding quantum heat flux due to the quantum nature of the KG equation. The second term on the right hand side of equation (34) would make the crucial difference between a classical and a quantum first law of thermodynamics.

Analogously, for the magnetic heat we have

$$\begin{aligned} \frac{\hat{d}Q_B}{dt} &= \int \nabla \cdot \mathbf{J}_B \, dV = \int \mathbf{v} \cdot (n \mathbf{j}_B) \, dV \\ &= \frac{m}{e} \int n \left(\frac{\partial \mathbf{A}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{A} + (\mathbf{A} \cdot \nabla) \mathbf{v} \right) \cdot \mathbf{v} \, dV \end{aligned} \quad (35)$$

where the vector potential \mathbf{A} fulfills the Maxwell equations, in terms of the fluxes (16) it reads

$$F^{\mu\nu}{}_{,\nu} = -j^\mu \quad (36)$$

where as usual $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. In terms of the vector and the electric potential, the Maxwell equations are given by

$$\square \mathbf{A} = -\mathbf{j} \quad (37a)$$

$$\square \varphi = -j - \frac{2en}{\kappa^2} \hat{m} \quad (37b)$$

where we have used the Lorentz gauge. Observe that the fluxes contain the information of the velocity of the fluid and of the electromagnetic term as well. This point will be important for the superconductivity.

THE CONDENSATION TEMPERATURE

First, we analyze the condensation temperature in the ideal case. In this situation the associated single-particle dispersion relation is given by

$$E^2 = (pc)^2 + (mc^2)^2. \quad (38)$$

In order to compare our case with well-know critical condensation temperatures, we start analyzing the ultra-relativistic and the non relativistic cases. For these we can express the dispersion relation (38) as $E \sim p^s$ [40], where $s = 1$ corresponds to the ultra-relativistic system and $s = 2$ stands for the non-relativistic case (these topics have been extensively studied, see for example [34–39] and references therein). Taken into account the number of antiparticles, in the ultra-relativistic limit the condensation temperature is given by

$$k_B T_c = \left(\frac{3\hbar^3 c n}{m} \right)^{1/2}, \quad (39)$$

where n is the charge density $n = \frac{N-\bar{N}}{V}$, N is the number of particles, \bar{N} corresponds to the number of anti-particles and V is the volume of the system. The quantity $N - \bar{N}$ can be written as

$$N - \bar{N} = \sum_{\mathbf{p}} [n_{\mathbf{p}}(\mu, T) - \bar{n}_{\mathbf{p}}(\mu, T)], \quad (40)$$

where n obeys the Bose–Einstein distribution function,

$$n_{\mathbf{p}} = \frac{1}{e^{\beta(E-\mu)} - 1}. \quad (41)$$

Here μ is the chemical potential and $\beta = 1/k_B T$. Similarly for the anti-bosons \bar{n} we have

$$\bar{n}_{\mathbf{p}} = \frac{1}{e^{\beta(E+\mu)} - 1}. \quad (42)$$

The chemical potential is bounded as $|\mu| \leq mc^2$ and at the condensation temperature we have that $|\mu| = mc^2$, which corresponds to the minimum of the associated energy. For ultra-relativistic systems without antiparticles [41], the corresponding condensation temperature is given by

$$k_B T_c = \left(\frac{\pi^2 N}{V \zeta(3)} \right)^{1/3} \hbar c. \quad (43)$$

Finally, in the non-relativistic case [40, 41]

$$k_B T_c = \frac{2\pi\hbar^2}{m} \left(\frac{N}{V \zeta(3/2)} \right)^{2/3}, \quad (44)$$

where $\zeta(x)$ denotes the Riemann zeta function. Expressions (39), (43) and (44) corresponds to 3-dimensional systems, but these expressions can be generalized to different dimensions.

On the other hand, the analysis of ideal and weakly interacting non-relativistic Bose–Einstein condensates with a finite number of particles, trapped in different potentials (see [8, 20–33, 40–42] and references therein) shows that the main properties associated with the condensate, in particular the condensation temperature, strongly depends on the characteristics of the trapping potential, the number of spatial dimensions and the associated single-particle energy spectrum.

In what follows we obtain the condensation temperature associated with our system within the semiclassical approximation [8, 20, 42]. Inserting plane waves in the KG equation (7), neglecting the term proportional to T^4 in (6), assuming that the temperature is sufficiently small and considering the low velocities limit, allows us to obtain the single-particle dispersion relation between energy and momentum [19]

$$E_p \simeq \frac{p^2}{2m} + \frac{\lambda}{2mc^2} |\Phi|^2 + \frac{\lambda}{4mc^2} (k_B T)^2 + mc^2 \phi + \frac{e^2 A^2}{2mc^2}. \quad (45)$$

We remain that in this work we interpret $\kappa^2 |\Phi(\vec{r}, t)|^2$ as the spatial density $n(\vec{r}, t)$ of the cloud, being κ the *scale* of the system. Notice that if we set $\lambda = 8\pi\hbar^2 c^2 a \kappa^2$ and $\phi = \alpha r^2$ in (45), where $\alpha = 1/2(\omega_0/c)^2$ and ω_0 is a frequency, we obtain the semiclassical energy spectrum in the Hartree–Fock approximation for a bosonic gas trapped in an isotropic harmonic oscillator [8, 20, 42], but with two extra terms due to the contributions of the thermal bath and to the electromagnetic field.

In this work we consider only two cases, the first one is for A^2 proportional to a constant. As we will see, in this case the associated constant can be absorbed by the chemical potential μ . For the second case we consider a dependence of the form $A \sim r^2$.

Assuming static thermal equilibrium $n(\vec{r}, t) \approx n(\vec{r})$ [20], thus

$$|\Phi|^2 \equiv \kappa^{-2} n(\vec{r}). \quad (46)$$

The spatial density within the semiclassical approximation reads [8, 20]

$$n(\vec{r}) = \frac{1}{(2\pi\hbar)^3} \int d^3\vec{p} n(\vec{r}, \vec{p}), \quad (47)$$

where $n(\vec{r}, \vec{p})$ is the Bose–Einstein distribution function given by [8, 20]

$$n(\vec{r}, \vec{p}) = \frac{1}{e^{\beta(E_p - \mu)} - 1}. \quad (48)$$

The number of particles in the 3-dimensional space obeys the normalization condition [8, 20]

$$N = \int d^3\vec{r} n(\vec{r}). \quad (49)$$

Integrating (47) over the momentum space allows to obtain the spatial density associated to this system

$$n(\vec{r}) = \left(\frac{mk_B T}{2\pi\hbar^2} \right)^{3/2} g_{3/2}(Z) \quad (50)$$

where $Z = \exp[\beta(\mu - \frac{\lambda\kappa^{-2}}{2mc^2} n(\vec{r}) - \frac{\lambda(k_B T)^2}{4mc^2} - mc^2 \phi - \frac{e^2 A^2}{2mc^2})]$. The function $g_\nu(z)$ is the so-called Bose–Einstein function defined by [40]

$$g_\nu(z) = \frac{1}{\Gamma(\nu)} \int_0^\infty \frac{x^{\nu-1} dx}{z^{-1} e^x - 1}. \quad (51)$$

being $\Gamma(\nu)$ the Gamma function. In order to calculate the condensation temperature let us suppose that our gas is trapped in a harmonic oscillator type-potential $\phi \sim r^2$. Clearly, this can be extended to a more general potentials. Expanding (50) at first order in the coupling constant λ , using the properties of the Bose–Einstein functions [40], allows us to express the spatial density as follows

$$n(\vec{r}) \approx n_0(\vec{r}) - \lambda g_{3/2}(z(\vec{r})) \left[\frac{\Lambda^{-6} \kappa^{-2}}{2mc^2 \kappa_B T} g_{1/2}(z(\vec{r})) + \Lambda^{-3} \frac{\kappa_B T}{4mc^2} \frac{g_{1/2}(z(\vec{r}))}{g_{3/2}(z(\vec{r}))} \right], \quad (52)$$

where

$$n_0(\vec{r}) = \Lambda^{-3} g_{3/2}(z(\vec{r})), \quad (53)$$

is the density for the case $\lambda = 0$, $\Lambda = (2\pi\hbar^2/m\kappa_B T)^{1/2}$ is the thermal de Broglie wavelength, and $z(\vec{r}) = \exp(\beta(\mu - \alpha mc^2 r^2 - e^2 A^2/2mc^2))$. In the case $A^2 \sim r^2$, with the help of the normalization condition (49) we obtain

$$\begin{aligned} N &\simeq \left(\frac{m}{2\Omega\hbar^2} \right)^{3/2} (k_B T)^3 g_3(e^{\beta\mu}) \\ &- \frac{\lambda \kappa^{-2} m^2 (k_B T)^{7/2}}{16\pi^{3/2} c^2 \hbar^6 \Omega^{3/2}} G_{3/2}(e^{\beta\mu}) \\ &- \frac{\lambda}{4c^2} \left(\frac{m^{1/3}}{2\Omega\hbar^2} \right)^{3/2} (k_B T)^4 g_2(e^{\beta\mu}), \end{aligned} \quad (54)$$

where

$$G_{3/2}(e^{\beta\mu}) = \sum_{i,j=1}^{\infty} \frac{e^{(i+j)\beta\mu}}{i^{1/2} j^{3/2} (i+j)^{3/2}}, \quad (55)$$

and $\Omega = \alpha mc^2 + \text{const } e^2/2mc^2$. From expression (54), we easily obtain the case $A^2 = \text{const}$,

$$\begin{aligned} N &\simeq \left(\frac{1}{2\alpha c^2 \hbar^2} \right)^{3/2} (k_B T)^3 g_3(e^{\beta(\mu-\gamma)}) \\ &- \frac{\lambda \kappa^{-2}}{2c^5} \left(\frac{m^{1/6}}{2\pi^{1/2} \hbar^2 \alpha^{1/2}} \right)^3 (k_B T)^{7/2} G_{3/2}(e^{\beta(\mu-\gamma)}) \\ &- \frac{\lambda}{4mc^5} \left(\frac{1}{2\alpha \hbar^2} \right)^{3/2} (k_B T)^4 g_2(e^{\beta(\mu-\gamma)}), \end{aligned} \quad (56)$$

where γ is defined as $\gamma = \text{const } e^2/2mc^2$. We notice immediately from the expressions (54) and (56) that if the function A^2 is position dependent, the correction in the number of particles can be associated to an effective external potential. On the other hand, when the function A^2 is constant, the correction can be associated to an effective chemical potential. If we set $\lambda = 0$ in expressions (54) and (56), we recover the expressions for the number of particles in the non-interacting case. In the thermodynamic limit, in the non-interacting case $\lambda = 0$, at the condensation temperature the value of the chemical potential is $\mu = 0$ [8]. If we further assume that above the condensation temperature the number of particles in the ground state is negligible, this allows us to obtain an expression for the condensation temperature T_0 . For the case $A^2 \sim r^2$, we obtain

$$k_B T_0 = \left(\frac{N}{\zeta(3)}\right)^{1/3} \left(\frac{2\Omega\hbar^2}{m}\right)^{1/2}, \quad (57)$$

Analogously, for the case $A^2 = 0$

$$k_B T_0 = \left(\frac{N\sqrt{8\alpha^3}}{\zeta(3)}\right)^{1/3} \hbar c. \quad (58)$$

Otherwise, using the properties of the Bose-Einstein functions, particularly for $g_3(e^{-\gamma/k_B \tilde{T}_0})$, when $-\gamma/k_B \tilde{T}_0 \rightarrow 0$ [40], where \tilde{T}_0 is the condensation temperature associated to the case $A^2 = \text{const} \neq 0$, we obtain the shift in the condensation temperature respect to (58) caused by $\gamma \neq 0$, in function of the number of particles,

$$\frac{\tilde{T}_0 - T_0}{T_0} \approx \gamma \frac{\zeta(2)}{3\zeta(3)^{2/3} \hbar c \sqrt{8\alpha^3}} N^{-1/3}. \quad (59)$$

Clearly, the shift (59) vanishes when the number of particles $N \rightarrow \infty$ and tends to the value (58). In order to obtain the leading correction in the shift for the condensation temperature caused by the coupling constant λ and the thermal bath in our system, let us expand the expressions (54) and (56) at first order in $T = T_0$, $\mu = 0$, $\lambda = 0$, and $\gamma = 0$. Additionally, at the condensation temperature, the chemical potential in the semiclassical approximation is given by $\mu_c = \frac{\lambda\kappa^{-2}}{2mc^2} n(\vec{r} = 0)$, as it is suggested from expression (50), thus

$$\begin{aligned} \mu_c \approx & \frac{\lambda\kappa^{-2} m^{1/2} (\kappa_B T_c)^{3/2} \zeta(3/2)}{2(2\pi)^{3/2} c^2 \hbar^3} \\ & - \lambda^{3/2} \frac{\sqrt{2\pi} \kappa^{-2} (\kappa_B T_c)^2}{(2\pi c^2 \hbar^2)^{3/2}}. \end{aligned} \quad (60)$$

Expression (60) basically corresponds to the definition of the chemical potential at the condensation temperature in the usual case [8, 20], except for the extra term contribution due to the thermal bath, and comes from the fact that $g_{3/2}(e^{-\delta}) \approx \zeta(3/2) - |\Gamma(-1/2)|\delta^{1/2}$, when $\delta \rightarrow 0$

[40]. Using these facts, we finally obtain the shift in the condensation temperature caused by λ and the thermal bath, in function of the number of particles in the case $A^2 \sim r^2$

$$\frac{T_c - T_0}{T_0} \equiv \frac{\Delta T_c}{T_0} = -\lambda \frac{m^{1/2}}{\kappa^2 \hbar^3 c^2} \Theta_1 \Xi N^{1/6} + \lambda \Theta_2 \Xi^2 N^{1/3}, \quad (61)$$

where

$$\Theta_1 = \frac{1}{3\zeta(3)} \left(\frac{\zeta(3/2)\zeta(2)}{2(2\pi)^{3/2}} - G_{3/2}(1) \right),$$

$$\Theta_2 = \frac{1}{3\zeta(3)} \left(\frac{1}{4mc^2} + \frac{(2\lambda)^{1/2} \zeta(2)\pi}{(2\pi)^{3/2} \kappa^2 \hbar^3 c^3} \right)$$

and $\Xi = (2\Omega\hbar^2/m)^{1/4}$ with T_0 defined in (57). A similar analysis, leads us to the shift in the condensation temperature caused by the coupling constant and the thermal bath associated with the case $A^2 = \text{const}$

$$\begin{aligned} \frac{T_c - T_0}{T_0} \equiv \frac{\Delta T_c}{T_0} = & -\lambda \frac{m^{1/2}}{\kappa^2 \hbar^3 c^2} \Theta_1 \tilde{\Xi} N^{1/6} + \lambda \Theta_2 \tilde{\Xi}^2 N^{1/3} \\ & + \gamma \frac{\zeta(2)}{3\zeta(3)^{2/3} \hbar c \sqrt{8\alpha^3}} N^{-1/3}, \end{aligned} \quad (62)$$

with $\tilde{\Xi} = (2\alpha c^2 \hbar^2)^{1/4}$ and T_0 defined in (58). In the case $\lambda = 0$ we recover from (62) the shift (59), as expected. From (61) and (62) we observe that the condensation temperature T_c is corrected with respect to the usual case T_0 as a consequence of the thermal bath and the contribution of the field A^2 , included in the semiclassical energy spectrum (45). Additionally, we notice that the critical temperature associated to the symmetry breaking (10) becomes very large when $\lambda \rightarrow 0$ and the condensation temperatures (61) and (62) tend to the non-interacting values (57) and (59) respectively. Setting $\alpha = 1/2(\omega_0/c)^2$ and $\lambda = 8\pi\hbar^2 c^2 \kappa^2 a$ into expressions (61) and (62) we recover the condensation temperature for a bosonic gas trapped in an isotropic harmonic oscillator potential, but corrected by the contributions of the thermal bath and the external field A^2 . For the sake of simplicity, let us analyze the correction over the usual result caused by the thermal bath and the external field $A^2 = \text{const}$ in (62). For instance, in the case of ^{87}Rb , with $a \sim 10^{-9}\text{m}$, $N \sim 10^6$, and $\omega \sim 10\text{Hz}$, we obtain from the second right hand term in (62) a correction up to $7.9 \times 10^{-78} \kappa^2 + 7.5 \times 10^{-38} \kappa$, and for the third right hand term, which is independent of the *scale* κ , up to $\text{const} \times 10^2$. In other words, the *scale* κ must be very large (up to 10^{38}) and the external field A^2 must be very weak (smaller than, or of the order of 10^{-12}), at least near to the center of the system, in order to obtain relevant corrections over the usual result under typical conditions. For these values the symmetry breaking temperature is $\sim 10^6\text{K}$. Finally, with the experimental data

given above, we obtain for the first right hand side term of expression (62) the usual shift $\sim 10^{-2}$, as expected [20]. The temperatures T_c^{SB} and T_c (or more specifically, $\Delta T_c/T_0$) are related through the coupling constant λ , this fact could be used as a criterium to compare both temperatures and in principle, to infer bounds related to the scale κ .

THE PHASE TRANSITION

From hereafter we study the transition between the $\Phi = 0$ state to the minimum $R_{min} = k_B/2\sqrt{(T_c^{SB})^2 - T^2}$ with $T < T_c^{SB}$.

During the time when $T \gg T_c^{SB}$ there are not scalar particles in the symmetry broken state. Below the critical temperature $T < T_c^{SB}$, close to the local minimum the density oscillates around the value $n = k_B^2 \kappa^2 ((T_c^{SB})^2 - T^2)/4$. We study the case when the function S in (14) has the simple expression, $S = s_0 t$, with $s_0 \ll mc/\hbar$ in the non-relativistic limit. This implies that the velocity $\mathbf{v} = 0$ as well. If there does not exist an external force in the system, then $\mathbf{F}_\phi = 0$. In this case the viscosity (dissipative term) of the BEC might in fact contain the whole information of the phase transition. From equation (23) we observe that the viscosity $\nabla\sigma$ contains only a term with the anisotropies of the temperature. That means that when the temperature of the system isotropies the fluid becomes a superfluid. Furthermore, from the expression for flux (16a) we observe that the vectorial flux contains only a term with the vector potential \mathbf{A} . Thus, the flux expression (16a) becomes the London equation,

$$\mathbf{j} = \frac{2ne^2}{\kappa^2} \mathbf{A}$$

indicating that the system becomes superconductor. The Maxwell equation (37a) becomes

$$\square \mathbf{A} = \frac{2ne^2}{\kappa^2} \mathbf{A}$$

which is the Proca equation, indicating that the photon acquires a mass $2ne^2/\kappa^2 c$. Obviously, playing with the conditions of the system we can find situations with superconductivity or superfluidity in different situations.

Finally, to illustrate the previous exposition, we give the following example. Suppose that in the system there are only condensed and excited particles of the same specie. We interpret $n = \kappa^2 k_B^2 ((T_c^{SB})^2 - T^2)/4$ as the particles density when the symmetry $U(1)$ has been broken. At $T = 0$ we expect that all particles have passed from the symmetry state into the broken symmetry one, such that the total particle density in the system is $n_T = \kappa^2 k_B^2 (T_c^{SB})^2/4$. Thus, in any time the number of particles N_0 in symmetry broken state is given by

$$N_0 = N \left[1 - \left(\frac{T}{T_c^{SB}} \right)^2 \right], \quad (63)$$

where $n/n_T = N_0/N$. Note that in this case the exponent 2 in the critical temperature appears naturally. As usual this expression shows the dependence of the condensate fraction N_0/N as a smooth function of temperature from $T \sim T_c^{SB}$ down into $T = 0$. In this case, the finite temperature terms are obtained from the one loop corrections of the SF density and are in complete agreement with the standard theory, [35, 38, 39].

Observe that the total particle density $n_T = \kappa^2 k_B^2 (T_c^{SB})^2/4$ of the scalar particles reaching the symmetry broken state at $T = 0$ can only be determined experimentally and fix the value of the scale κ . So in principle we are able to mimic the result that in the presence of interactions we have $N_0 < N$. The main idea we want to point out here is that these phenomena might be equivalent for a BEC on earth as well as for cosmological scales, so this model might be tested in the laboratory. If confirmed, the phase transition of a BEC can be explained using quantum field theory in a straightforward way.

CONCLUSIONS

In this work we studied the phase transition of a boson gas with zero spin represented by the KG equation with a Lagrangian containing a $U(1)$ symmetry, with mass m and self-interaction λ , given by a mexican hat SF potential, immersed in a thermal bath at temperature T , close to the critical temperature of symmetry breaking, up to one loop in perturbations theory. We rewrite the KG equation and interpret it as a generalized Gross-Pitaevskii one at finite temperature. We show that the transition from the phase with the $U(1)$ symmetry to the phase with this symmetry broken can be interpreted as a phase transition from the gas state to the condensation state of the Bose gas. We obtained the condensation temperature as well. By rewriting the generalized Gross-Pitaevskii equation in terms of hydrodynamic quantities we were able to derive the thermodynamic of the phase transition, with viscosity and dissipative terms and find that the first law of the thermodynamics contains a new term that is a direct consequence of the quantum character of KG equation. The main result of the present work is that the phase transition do not take place at the same temperature and conditions of the condensation. We saw that, for example, Bose-Einstein condensation take place in a ^{87}Rb crystal in normal density conditions, but that the phase transition can take place only in very high density conditions, for example, at densities like in a neutron star. On the other hand, other materials have phase transitions and/or condensations in different conditions of density and temperature, depending on their mass, self-interaction parameter λ and the value of the scale κ . We gave different examples.

It remains to see whether this generalization of the

Gross-Pitaevskii equation can describe the transition of a Bose gas into a BEC state in the laboratory. In other words, we propose that the superfluid and/or superconductor like-behavior in a BEC can be measured experimentally in a laboratory, in order to compare the results given here with realistic systems.

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- [1] M. R. Anderson et al, Science 269, 198–201, (1995).
- [2] Ph.W. Courteille, V.S. Bagnato, V.I. Yukalov, I. Laser Phys., 11, 659 (2001).
- [3] M. B. Pinto, R. O. Ramos and J. E. Parreira, Phys. Rev. D 71 123519, (2005).
- [4] L. Dolan and R. Jackiw, Phys. Rev. D 9, 3320 (1974).
- [5] S. Weinberg, Phys. Rev. Lett. 9 3357 (1974).
- [6] T. Matos and A. Suarez, Europhys. Lett. **96** (2011) 56005 [arXiv:1110.3114 [gr-qc]].
- [7] T. Matos and A. Suarez, arXiv:1103.5731 [gr-qc].
- [8] C. J. Pethick and H. Smith, *Bose-Einstein Condensation in Diluted Gases*, Cambridge University Press, Cambridge (2006).
- [9] E. Castellanos and T. Matos, arXiv:1202.3793.
- [10] Dany Page, James M. Lattimer, Madappa Prakash Andrew W. Steiner ApJ 155, 623, (2004)
- [11] Juan Magana, Tonatiuh Matos, Victor Robles, Abril Suarez. A brief Review of the Scalar Field Dark Matter model. J.Phys.Conf.Ser. 378 (2012) 012012, arXiv:1201.6107
- [12] O. Erken, P. Sikivie, H. Tam, and Q. Yang. Phys. Rev. Lett. **108**, 061304 (2012).
- [13] Juan Barranco Argelia Bernal. Phys. Rev. D **83**, 043525 (2011).
- [14] Victor H. Robles, T. Matos. Exact Solution to Finite Temperature SFDM: Natural Cores without Feedback. arXiv:1207.5858
- [15] T. Chiueh, Phys. Rev. E 57, 4150 (1998).
- [16] D. Bohm, Phys. Rev., 85 (1952) 180.
- [17] G. Grossing, Phys. Lett. A, 388 (2009) 811.
- [18] Oliver, X. and de Saracibar, C. A., *Mecánica de Medios Continuos*, edited by Vieira, E. and Car E. (UPC) 2000.
- [19] B. Zwiebach, Cambridge University Press, Cambridge (2004).
- [20] F. Dalfovo, S. Giordini, L. Pitaevskii, S. Stringari, Reviews of Modern Physics, Vol. 71, No. 3, April (1999) pp. 463-512.
- [21] Ketterle, W., and N. J. van Druten, Phys. Rev. A **54** (1996b).
- [22] Bagnato, V., D. E. Pritchard, and D. Kleppner, Phys. Rev. A **35** (1987).
- [23] S. Grossmann and M. Holthaus, Phys. Lett. A **208** (1995).
- [24] Giorgini, S., L. Pitaevskii, and S. Stringari, Phys. Rev. A **54** (1996)
- [25] H. Haugerud, T. Haugset, F. Ravnal, Phys. Lett. A **225** (1997).
- [26] H. Shi and W. M. Zheng, Phys. Rev. A **56** 1046, (1997)
- [27] Z. Yan, Phys. A **298**, 455 (2001)
- [28] L. Salasnich, Int. J. Mod. Phys. B **16**, 2185 (2002)
- [29] O. Zobay, J. Phys. B 37, 2593 (2004).
- [30] V. I. Yukalov, Laser Phys. Lett. 1, 435-461 (2004)
- [31] V. I. Yukalov, Phys. Rev. A **72**, 033608 (2005).
- [32] A. Jaouadi, M. Telmini, and E. Charron, arXiv:1011.6477v1 [cond-mat.quant-gas] (2010).
- [33] V.I. Yukalov, Phys. Part. Nucl. 42 (2011) 460-513.
- [34] F. Jutter, Z. Phys. 47, 542 (1928).
- [35] H. E. Haber and H. A. Weldon, Phys. Rev. Lett. 46, (1981)
- [36] H. E. Haber and H. A. Weldon, Phys. Rev. D **25**, 502 (1982).
- [37] M. Grether, M. de Llano, and G. A. Baker, Phys. Rev. Lett. 99, 200406 (2007).
- [38] S. Singh and P. N. Pandita, Phys. Rev. A **28**, 1752 (1983)
- [39] S. Singh and R. K. Pathria, Phys. Rev. A **30**, 442 (1984).
- [40] R. K. Pathria, *Statistical Mechanics*, Butterworth Heinemann, Oxford (1996).
- [41] W. Greiner and Neise Stocker, *Thermodynamics and Statistical Mechanics*, Springer-Verlag, New York (1995).
- [42] Pitaevskii, L. Stringari, S. Bose-Einstein Condensation. Ed. Birman, J. et. al Clarendon Press, Oxford, (2003).